Entanglement degradation of a two-mode squeezed vacuum in absorbing and amplifying optical fibers

Stefan Scheel[1], Tomaš Opatrný[1,2], and Dirk-Gunnar Welsch[1]

[1]Theoretisch-Physikalisches Institut, Friedrich-Schiller-Universität Jena, Max-Wien-Platz 1, D-07743 Jena, Germany

[2]Department of Theoretical Physics, Palacký University, Svobody 26, 771 46 Olomouc, Czech Republic

Abstract

Applying the recently developed formalism of quantum-state transformation at absorbing dielectric four-port devices [L. Knöll, S. Scheel, E. Schmidt, D.-G. Welsch, and A.V. Chizhov, Phys. Rev. A **59**, 4716 (1999)], we calculate the quantum state of the outgoing modes of a two-mode squeezed vacuum transmitted through optical fibers of given extinction coefficients. Using the Peres–Horodecki separability criterion for continuous variable systems [R. Simon, Phys. Rev. Lett. **84**, 2726 (2000)], we compute the maximal length of transmission of a two-mode squeezed vacuum through an absorbing system for which the transmitted state is still inseparable. Further, we calculate the maximal gain for which inseparability can be observed in an amplifying setup. Finally, we estimate an upper bound of the entanglement preserved after transmission through an absorbing system. The results show that the characteristic length of entanglement degradation drastically decreases with increasing strength of squeezing.

1 Introduction

Recently there has been increasing interest in the use of entangled continuous variable systems in quantum communication [1]. The most prominent example is the two-mode squeezed vacuum which in the Fock basis reads as

$$|\psi\rangle = \mathrm{e}^{\zeta\left(\hat{a}_1^\dagger \hat{a}_2^\dagger - \hat{a}_1\hat{a}_2\right)}|00\rangle = \sqrt{1-q^2}\sum_{n=0}^{\infty} q^n\,|nn\rangle \tag{1}$$

[$q=\tanh\zeta$, ζ real] and whose Wigner function is a Gaussian. When transmitted through a noisy channel, such as optical fibers of given extinction coefficients, the resulting state will in general be mixed. For further use of the state in some quantum communication experiment the question of the entanglement that is available after the transmission arises. Real entanglement measures, however, are difficult to compute and are mostly known only numerically. A typical example is the entanglement measure based on the relative entropy where the distance of the density matrix of the state under consideration to the set of all separable density matrices must be computed [2]. In practice, this becomes impossible for states like a two-mode squeezed vacuum with reasonable strength of squeezing, because for comparable numerical accuracy the Hilbert space can only be truncated at higher dimension with increasing squeezing strength.

Here, we proceed in a different direction. Starting from the quantum-optical input-output relations of light at absorbing dielectric four-port devices [3], we first present the basic formulas for determining the output quantum state from the input quantum state and the characteristic transmission and absorption matrices of the devices (Sec. 2). We then apply the input-output formalism to the calculation of the output Wigner function observed when two modes that are initially prepared in a squeezed vacuum are transmitted through absorbing fibers. Using the Peres-Horodecki separability criterion for Gaussian states [4], we calculate the maximal length of transmission for which the output state is still inseparable in principle (Sec. 3). Calculating the density matrix of the transmitted light in the Fock basis, on applying the formalism developed in [5], we finally extract some pure state from the output density matrix, use the convexity property of the entanglement measure [6], and derive an estimate of the amount of entanglement available after transmission (Sec. 4).

2 Quantum-state transformation

Let us first consider the quantum-optical input-output relations and the corresponding quantum-state transformation formulas for light at dispersing and absorbing dielectric four-port devices. We restrict ourselves to a quasi one-dimensional scheme, as

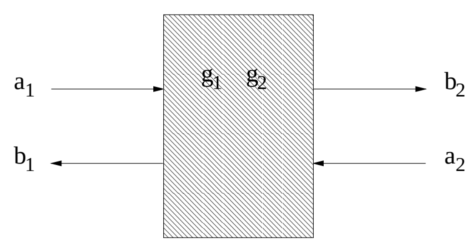

Figure 1: Scheme of the dielectric four-port device, with $\hat{a}_i = \hat{a}_i(\omega)$ $[\hat{b}_i = \hat{b}_i(\omega)]$ and $\hat{g}_i = \hat{g}_i(\omega)$ being the destruction operators of the input [output] photons and the device excitations respectively.

depicted in Fig. 1, in which the dielectric device is surrounded by vacuum. Applying the formalism developed in [7], we quantize the electromagnetic field in the presence of the device by means of a Green function representation of the field and introduction of bosonic fields playing the role of the collective excitations of the field, the dielectric matter, and the reservoir. It turns out that outside the device the usual mode expansion applies, with the $\hat{a}_i(\omega)$ and $\hat{b}_i(\omega)$ in Fig. 1 being respectively the photonic operators of the incoming and outgoing plane waves at frequency ω. The $\hat{g}_i(\omega)$ in the figure are the bosonic operators of the device excitations and play the role of noise forces associated which absorption. It then follows that the action of the dielectric device on the incoming radiation can be described by quantum optical input-output relations which, in fact, are nothing but a suitable rewriting of the corresponding one-dimensional Green function. Let us introduce the compact two-vector notation $\hat{\mathbf{a}}(\omega)$, $\hat{\mathbf{b}}(\omega)$, and $\hat{\mathbf{g}}(\omega)$ for the field and device operators respectively. The input-output relations can then be written in the compact form [3]

$$\hat{\mathbf{b}}(\omega) = \mathbf{T}(\omega)\hat{\mathbf{a}}(\omega) + \mathbf{A}(\omega)\hat{\mathbf{g}}(\omega), \tag{2}$$

where the characteristic transmission and absorption matrices $\mathbf{T}(\omega)$ and $\mathbf{A}(\omega)$, respectively, satisfy the energy-conservation relation

$$\mathbf{T}(\omega)\mathbf{T}^{+}(\omega) + \mathbf{A}(\omega)\mathbf{A}^{+}(\omega) = \mathbf{I}. \tag{3}$$

Equations (2) and (3) are valid for any frequency. They allows us to construct the output operators from the input operators over the whole frequency range.

The second step in the quantum-state transformation corresponds to an open-systems approach. Suppose the incoming field is prepared in some state of the Hilbert space $\mathcal{H}_{\text{field}}$ and the device (including the reservoir) is initially prepared in some state of the Hilbert space $\mathcal{H}_{\text{device}}$. In the full Hilbert space, which is the tensor product $\mathcal{H}_{\text{field}} \otimes \mathcal{H}_{\text{device}}$, a unitary operator transformation can then be constructed, whereas in the space $\mathcal{H}_{\text{field}}$ it could not due to the dissipation processes. Let us define the four-vector operators

$$\hat{\boldsymbol{\alpha}}(\omega) = \begin{pmatrix} \hat{\mathbf{a}}(\omega) \\ \hat{\mathbf{g}}(\omega) \end{pmatrix}, \quad \hat{\boldsymbol{\beta}}(\omega) = \begin{pmatrix} \hat{\mathbf{b}}(\omega) \\ \hat{\mathbf{h}}(\omega) \end{pmatrix}, \tag{4}$$

where $\hat{\mathbf{h}}(\omega)$ is some auxiliary (two-vector) bosonic device operator. Then, the input-output relation (2) can be extended to a four-dimensional transformation

$$\hat{\boldsymbol{\beta}}(\omega) = \boldsymbol{\Lambda}(\omega)\hat{\boldsymbol{\alpha}}(\omega)\,, \tag{5}$$

with $\boldsymbol{\Lambda}(\omega) \in \mathrm{SU}(4)$ [5]. Explicitly,

$$\boldsymbol{\Lambda}(\omega) = \begin{pmatrix} \mathbf{T}(\omega) & \mathbf{A}(\omega) \\ -\mathbf{S}(\omega)\mathbf{C}^{-1}(\omega)\mathbf{T}(\omega) & \mathbf{C}(\omega)\mathbf{S}^{-1}(\omega)\mathbf{A}(\omega) \end{pmatrix} \tag{6}$$

with the commuting positive Hermitian matrices

$$\mathbf{C}(\omega) = \sqrt{\mathbf{T}(\omega)\mathbf{T}^{+}(\omega)}\,, \quad \mathbf{S}(\omega) = \sqrt{\mathbf{A}(\omega)\mathbf{A}^{+}(\omega)}\,. \tag{7}$$

Hence, there is a unitary operator transformation

$$\hat{\boldsymbol{\beta}}(\omega) = \hat{U}^{\dagger}\hat{\boldsymbol{\alpha}}(\omega)\hat{U} \tag{8}$$

where

$$\hat{U} = \exp\left\{-i\int \mathrm{d}\omega\,\left[\hat{\boldsymbol{\alpha}}^{\dagger}(\omega)\right]^{T}\boldsymbol{\Phi}(\omega)\hat{\boldsymbol{\alpha}}(\omega)\right\} \tag{9}$$

and

$$\mathbf{\Lambda}(\omega) = \mathrm{e}^{-i\mathbf{\Phi}(\omega)}\,. \tag{10}$$

Note that $\hat{U}$ acts in the product space $\mathcal{H}_{\mathrm{field}} \otimes \mathcal{H}_{\mathrm{device}}$. Given a density operator $\hat{\varrho}_{\mathrm{in}}$ of the input quantum state as a functional of $\hat{\boldsymbol{\alpha}}(\omega)$, the density operator of the output quantum state is obtained by a unitary transformation with the operator $\hat{U}$ from Eq. (9) and projecting back onto the Hilbert space $\mathcal{H}_{\mathrm{field}}$. Hence,

$$\hat{\varrho}_{\mathrm{out}}^{(\mathrm{F})} = \mathrm{Tr}^{(\mathrm{D})}\left\{\hat{U}\hat{\varrho}_{\mathrm{in}}\hat{U}^{\dagger}\right\} = \mathrm{Tr}^{(\mathrm{D})}\left\{\hat{\varrho}_{\mathrm{in}}\left[\mathbf{\Lambda}^{+}(\omega)\hat{\boldsymbol{\alpha}}(\omega), \mathbf{\Lambda}^{T}(\omega)\hat{\boldsymbol{\alpha}}^{\dagger}(\omega)\right]\right\}\,, \tag{11}$$

where $\mathrm{Tr}^{(\mathrm{D})}$ means tracing with respect to the device variables. Note, that the difference to usually considered open-systems theories is provided by the fact that we actually know how the dissipative environment (e.g., a dispersing and absorbing fiber) acts on our quantum states.

Let us briefly comment on amplifying devices. In contrast to absorbing devices, we have now to insert the noise *creation* operators $\hat{g}_i^{\dagger}(\omega)$ into the input-output relation (2) [i.e., $\hat{\mathbf{g}}(\omega) \to \hat{\mathbf{g}}^{\dagger}(\omega)$]. The relation (3) then changes to

$$\mathbf{T}(\omega)\mathbf{T}^{+}(\omega) - \mathbf{A}(\omega)\mathbf{A}^{+}(\omega) = \mathbf{I}\,, \tag{12}$$

where $\mathbf{A}(\omega)$ plays the role of the gain matrix. Further, the 4×4-matrix $\mathbf{\Lambda}(\omega)$ becomes an element of the noncompact group SU(2,2).

3 Application of the Peres–Horodecki separability criterion

Let us consider a two-mode squeezed vacuum which is transmitted through a noisy communication channel such as two absorbing (amplifying) dielectric four-port devices as sketched in Fig. 2, the characteristic transmission and absorption (gain) matrices of the devices being $\mathbf{T}^{(i)}(\omega)$ and $\mathbf{A}^{(i)}(\omega)$ respectively ($i=1,2$). In Eq. (1),

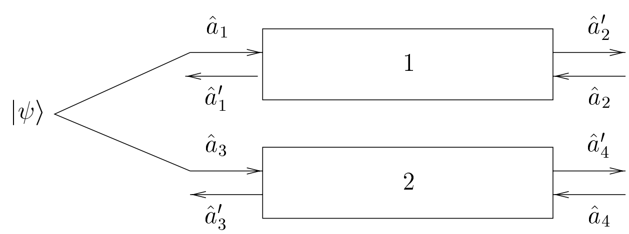

Figure 2: A two-mode input field prepared in the quantum state $|\psi\rangle$ is transmitted through two absorbing (amplifying) dielectric four-port devices, $\hat{a}_1, \hat{a}_3$ ($\hat{a}'_2, \hat{a}'_4$) being the photonic operators of the relevant input (output) modes.

the two-mode squeezed vacuum is given in the Fock basis. Equivalently, it can be expressed in terms of the Gaussian Wigner function

$$W(\boldsymbol{\xi}) = \left(4\pi^2\sqrt{\det \boldsymbol{V}}\right)^{-1} \exp\left\{-\tfrac{1}{2}\boldsymbol{\xi}^T \boldsymbol{V}^{-1}\boldsymbol{\xi}\right\}, \tag{13}$$

where $\boldsymbol{\xi}$ is a four-vector whose elements are the quadrature-component variables q_1, p_1, q_3, p_3, and $\boldsymbol{V}$ is the 4×4 variance matrix of the Wigner function,

$$\boldsymbol{V} = \begin{pmatrix} \mathbf{X} & \mathbf{Z} \\ \mathbf{Z}^T & \mathbf{Y} \end{pmatrix} = \begin{pmatrix} c/2 & 0 & -s/2 & 0 \\ 0 & c/2 & 0 & s/2 \\ -s/2 & 0 & c/2 & 0 \\ 0 & s/2 & 0 & c/2 \end{pmatrix} \tag{14}$$

($c = \cosh 2\zeta$, $s = \sinh 2\zeta$). Transmitting the two-mode squeezed vacuum through the four-port devices at some temperatures ϑ_i, the Wigner function of the transformed state is again a Gaussian. Using the input-output relations for absorbing (amplifying) devices as given in Sec. 2, we can easily transform the input variance matrix (14) to obtain the output variance matrix. The result is

$$\begin{aligned}
\langle \hat{a}_2'^{\dagger}\hat{a}_2'\rangle_s &= |T_{21}^{(1)}|^2\langle \hat{a}_1^{\dagger}\hat{a}_1\rangle_s + |T_{22}^{(1)}|^2\langle \hat{a}_2^{\dagger}\hat{a}_2\rangle_s \\
&\quad + |A_{21}^{(1)}|^2\langle \hat{g}_1^{(1)\dagger}\hat{g}_1^{(1)}\rangle_s + |A_{22}^{(1)}|^2\langle \hat{g}_2^{(1)\dagger}\hat{g}_2^{(1)}\rangle_s\,, \\
\langle \hat{a}_4'^{\dagger}\hat{a}_4'\rangle_s &= |T_{21}^{(2)}|^2\langle \hat{a}_3^{\dagger}\hat{a}_3\rangle_s + |T_{22}^{(2)}|^2\langle \hat{a}_4^{\dagger}\hat{a}_4\rangle_s \\
&\quad + |A_{21}^{(2)}|^2\langle \hat{g}_1^{(2)\dagger}\hat{g}_1^{(2)}\rangle_s + |A_{22}^{(2)}|^2\langle \hat{g}_2^{(2)\dagger}\hat{g}_2^{(2)}\rangle_s\,, \\
\langle \hat{a}_2'\hat{a}_4'\rangle &= T_{21}^{(1)}T_{21}^{(2)}\langle \hat{a}_1\hat{a}_3\rangle
\end{aligned} \tag{15}$$

(the subscript s refers to symmetric operator ordering as required for the Wigner function) which can also be rewritten as correlations of the quadratures. Introducing the abbreviating notation $T_{21}^{(i)} \equiv T_i$ and $T_{22}^{(i)} \equiv R_i$, we arrive at the following elements of the output variance matrix:

$$X_{11} = X_{22} = \tfrac{1}{2}c|T_1|^2 + \tfrac{1}{2}|R_1|^2 + \sigma\left(n_{\mathrm{th}\,1} + \tfrac{1}{2}\right)\left(1 - |T_1|^2 - |R_1|^2\right), \tag{16}$$

$$Y_{11} = Y_{22} = \tfrac{1}{2}c|T_2|^2 + \tfrac{1}{2}|R_2|^2 + \sigma\left(n_{\mathrm{th}\,2} + \tfrac{1}{2}\right)\left(1 - |T_2|^2 - |R_2|^2\right), \tag{17}$$

$$Z_{11} = -Z_{22} = -\tfrac{1}{2}s\mathrm{Re}\,(T_1T_2), \tag{18}$$

$$Z_{12} = Z_{21} = -\tfrac{1}{2}s\mathrm{Im}\,(T_1T_2) \tag{19}$$

($X_{12} = X_{21} = Y_{12} = Y_{21} = 0$), where $\sigma = +1(-1)$ for absorbing (amplifying) devices, and $n_{\mathrm{th}\,i} = [\exp \hbar\omega/(k_B\vartheta_i) - 1]^{-1}$ is the mean number of the (thermal) excitations of the ith device.

Application of the Peres–Horodecki separability criterion [4]

$$\det\mathbf{X}\det\mathbf{Y} + \left(\tfrac{1}{4} - |\det\mathbf{Z}|\right)^2 - \mathrm{Tr}\left(\mathbf{XJZJYJZ}^T\mathbf{J}\right) \geq \tfrac{1}{4}\left(\det\mathbf{X} + \det\mathbf{Y}\right) \tag{20}$$

with

$$\mathbf{J} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \tag{21}$$

to the output variance matrix yields (for equal devices) the inequality

$$n_{\rm th} \geq \frac{(1-\sigma)\,(1-|R|^2) + |T|^2\left(\sigma - {\rm e}^{-2|\zeta|}\right)}{2\sigma\,(1-|R|^2-|T|^2)}\,. \tag{22}$$

Hence, for chosen squeezing parameter ζ and chosen transmission and reflection coefficients T and R, respectively, there exists a maximal temperature and correspondingly a maximal mean excitation number of the thermal state in which each of the two devices is prepared such that the quantum state of the transmitted squeezed vacuum is still not separable.

In particular, for absorbing devices ($\sigma=+1$) the inequality (22) reads

$$n_{\rm th} \geq \frac{|T|^2\left(1 - {\rm e}^{-2|\zeta|}\right)}{2\,(1-|R|^2-|T|^2)}\,. \tag{23}$$

With regard to optical fibers with perfect input coupling, we may let $R=0$ and relate, according to the Lambert-Beer law, the transmission coefficient to the propagation length l as $|T|=e^{-l/l_{\rm A}}$, with $l_{\rm A}$ being the characteristic absorption length of the fibers. From the inequality (22) it then follows that the upper bound of the propagation length for which the transmitted squeezed vacuum is still not separable is

$$l = \tfrac{1}{2} l_{\rm A} \ln\left[1 + \frac{1}{2n_{\rm th}}\left(1 - {\rm e}^{-2|\zeta|}\right)\right]. \tag{24}$$

It is interesting to note that this result agrees with that calculated in [8], if the ‘renormalized time’ in [8] is replaced with $1-|T|^2$. Note that in our formalism the bound is simply obtained by using the quantum-optical input-output relations (2) for light at dispersing and absorbing dielectric four-port devices. Involved calculations of the dynamics of the Wigner function are not needed.

Moreover, the general result (22) also applies to amplifying devices ($\sigma=-1$). In particular, in the zero-temperature limit ($n_{\rm th}=0$) the boundary between inseparability and separability is reached when

$$|T|^2 = \frac{2\,(1-|R|^2)}{1+{\rm e}^{-2|\zeta|}} \tag{25}$$

is valid. For zero reflection ($R=0$), Eq. (25) reveals that the upper limit of the 'excess' gain $g=|T|^2-1\geq 0$ for which inseparability changes to separability is simply given by the squeezing parameter q,

$$g = |q| = \tanh|\zeta|\,. \tag{26}$$

An obvious consequence of Eq. (26) is that entanglement cannot be produced from the vacuum by amplification. For the vacuum the squeezing parameter has to be set equal to zero and thus from Eq. (26) it follows that any nonvanishing gain g must necessarily lead to a separable state. Another interesting fact is that there exists an absolute upper bound of the gain for which inseparability can be retained. Since the absolute value of the squeezing parameter (and thus the 'excess' gain g) is bounded by 1, one is left with the conclusion that an amplifier which doubles the intensity of a signal ($|T|^2=2$) destroys all but the maximal entanglement in a two-mode squeezed vacuum state which spoils the use of fiber amplifiers in quantum communication.

4 Entanglement estimates

The separability criterion exploited in Sec. 3 can tell us only if the transmitted quantum state is separable or not. It can not, however, provide us information about the amount of entanglement which is actually contained in the state. In order to obtain analytical estimates of the available amount of entanglement, we note that any entanglement measure (such as the distance of the state under consideration to the set of separable states measured by the relative entropy) has the convexity property [9]

$$E[\lambda\hat{\varrho}_1+(1-\lambda)\hat{\varrho}_2] \leq \lambda E(\hat{\varrho}_1) + (1-\lambda)E(\hat{\varrho}_2). \tag{27}$$

This property can advantageously be used to find bounds on the entanglement. If we are able to divide the quantum state into a sum of separable states (having no entanglement) and a single pure state, then an upper bound on the entanglement is given by the reduced von Neumann entropy of the extracted pure state [6].

Let us consider two modes that are initially prepared in a truncated version of the quantum state (1)

$$|\phi(q)\rangle = \frac{1}{\sqrt{1+q^2}}\left(|00\rangle + q|11\rangle\right), \tag{28}$$

which approximates a two-mode squeezed vacuum for small values of the squeezing parameter (i.e., $|q|\ll 1$). It is not difficult to prove that the entanglement of the state is

$$E = \ln\bigl(1+q^2\bigr) - \frac{q^2}{1+q^2}\,\ln q^2. \tag{29}$$

Applying the transformation formula (11), the quantum state in which the two modes are prepared after propagating through two fibers of transmission coefficients T_1 and T_2 is derived to be

$$\begin{aligned}\hat{\varrho}_{\mathrm{out}}^{(\mathrm{F})} = \frac{q^2}{1+q^2}\bigl[&\left(1-|T_1|^2\right)\left(1-|T_2|^2\right)|00\rangle\langle 00| + |T_1|^2\left(1-|T_2|^2\right)|10\rangle\langle 10| \\ &+ |T_2|^2\left(1-|T_1|^2\right)|01\rangle\langle 01|\bigr] + \frac{1+|q'|^2}{1+|q|^2}\,|\phi(q')\rangle\langle\phi(q')|,\end{aligned} \tag{30}$$

where

$$q' = T_1 T_2 q. \tag{31}$$

Here and the following we assume that the fibers are prepared in the ground state (low-temperature limit). Since the first term on the right-hand side in Eq. (30) is a sum of separable states and the second term is a pure state whose entanglement is given by Eq. (29) with q' in place of q, the entanglement of the state in Eq. (30) can be estimated, on recalling the convexity property (27), according to

$$E \leq \frac{1}{1+q^2}\left[\left(1+|qT_1T_2|^2\right)\ln\left(1+|qT_1T_2|^2\right) - |qT_1T_2|^2\ln|qT_1T_2|^2\right]. \tag{32}$$

We see that with increasing propagation lengths, i.e., with decreasing transmission coefficients, the entanglement of the transmitted light decreases more rapidly than the intensity.

Now let us return to the exact two-mode squeezed vacuum. Applying Eq. (11) and transforming the density operator $\hat{\varrho}_{\rm in} = |\psi\rangle\langle\psi|$ with $|\psi\rangle$ from Eq. (1), we derive after a lengthy, but straightforward calculation

$$\hat{\varrho}_{\rm out}^{\rm (F)} = \left(1-q^2\right) \sum_{m=0}^{\infty} \sum_{n=0}^{m} \frac{1}{m!n!} \left[\sum_{k=0}^{\infty} C_{m,n,k} \left(c_{m-n}|m-n+k\rangle\langle k| + \text{H.c.}\right)\right] \otimes \left[\sum_{l=0}^{\infty} D_{m,n,l} \left(d_{m-n}|m-n+l\rangle\langle l| + \text{H.c.}\right)\right], \tag{33}$$

where

$$C_{m,n,k} = \frac{q^n m!n! \left(1-|T_1|^2\right)^{n-k} |T_1|^{2k}}{(n-k!)\sqrt{k!(m-n-k!)}}, \tag{34}$$

$$c_{m-n} = q^{(m-n)/2} T_1^{m-n} \left(1-\tfrac{1}{2}\delta_{mn}\right) \tag{35}$$

$$D_{m,n,l} = \frac{q^n m!n! \left(1-|T_2|^2\right)^{n-l} |T_2|^{2l}}{(n-l!)\sqrt{l!(m-n-l!)}}, \tag{36}$$

$$d_{m-n} = q^{(m-n)/2} T_2^{m-n} \left(1-\tfrac{1}{2}\delta_{mn}\right). \tag{37}$$

Performing the sum over n then yields the final result

$$\hat{\varrho}_{\rm out}^{\rm (F)} = \left(1-q^2\right) \sum_{m=0}^{\infty} \sum_{k,l=0}^{\infty} K_{k,l,m} \left(c_m|m+k\rangle\langle k| + \text{H.c.}\right) \otimes \left(d_m|m+l\rangle\langle l| + \text{H.c.}\right), \tag{38}$$

where

$$K_{k,l,m} = \frac{[q^2(1-|T_1|^2)(1-|T_2|^2)]^a a!(a+m)!}{\sqrt{k!l!(k+m)!(l+m)!}(a-k)!(a-l)!} \left(\frac{|T_1|^2}{1-|T_1|^2}\right)^k \left(\frac{|T_2|^2}{1-|T_2|^2}\right)^l \times {}_3F_2\left[\begin{matrix} a+1, a+m+1, 1 \\ a-k+1, a-l+1 \end{matrix}; q^2\left(1-|T_1|^2\right)\left(1-|T_2|^2\right)\right] \tag{39}$$

[a=max(k,l)]. Note that one lower index of the hypergeometric function ${}_3F_2$ is always equal to unity, so that we effectively deal with a Gaussian hypergeometric function ${}_2F_1$.

We now try to decompose the density operator (38) into a pure state $|\Psi\rangle$ and some residual state $\hat{\varrho}'$ whose entanglement is desired to be sufficiently small,

$$\hat{\varrho}_{\rm out}^{\rm (F)} = \lambda\hat{\varrho}' + (1-\lambda)|\Psi\rangle\langle\Psi|, \tag{40}$$

A suitable pure state $|\Psi\rangle$ may be chosen such that

$$\sqrt{1-\lambda}\,|\Psi\rangle = \sqrt{\frac{1-q^2}{K_{000}}} \sum_{n=0}^{\infty} K_{00n} c_n d_n |nn\rangle. \tag{41}$$

It has the properties that (i) only matrix elements of the same type as in the initial squeezed vacuum occur and (ii) the coefficients of the matrix elements $|00\rangle \leftrightarrow |nn\rangle$ are met exactly, i.e.,

$$(1-\lambda)\langle 00|\Psi\rangle\langle\Psi|nn\rangle = \langle 00|\hat{\varrho}_{\rm out}^{(\rm F)}|nn\rangle. \tag{42}$$

This choice is of course not unique. Moreover, the residual state might still contain some entanglement. For small enough squeezing parameter q, however, the residual entanglement is expected to be small compared to the entanglement contained in the state (41), i.e., $\lambda E(\hat{\varrho}') \ll (1-\lambda)E(|\Psi\rangle)$. In principle, one can proceed and extract more and more pure states from the output state and apply the generalized inequality

$$E\Big(\sum_i p_i\hat{\varrho}_i\Big) \le \sum_i p_i\, E(\hat{\varrho}_i)\,, \qquad \sum_i p_i = 1. \tag{43}$$

Disregarding a possible (small) entanglement of the residual state $\hat{\varrho}'$, the entanglement of the pure state $|\Psi\rangle$ gives us some estimate of (the upper bound of) entanglement of the output state $\hat{\varrho}_{\rm out}^{(\rm F)}$,

$$\begin{aligned} E(\hat{\varrho}_{\rm out}^{(\rm F)}) &\approx (1-\lambda)\, E(|\Psi\rangle) \\ &= \frac{1-x}{(1-x)^2-y}\,\ln\!\left[\frac{1-x}{(1-x)^2-y}\right] + \frac{(1-x)\{[y+(1-x)^2]\ln(1-x)-y\ln y\}}{[y-(1-x)^2]^2}\,, \end{aligned} \tag{44}$$

where

$$x = q^2(1-|T_1|^2)(1-|T_2|^2), \tag{45}$$

$$y = |qT_1T_2|^2. \tag{46}$$

Note that for $T_1 = T_2 = 1$ Eq. (44) gives the correct entanglement of the input state,

$$E(\hat{\varrho}_{\rm out}^{(\rm F)})\Big|_{T_1=T_2=1} = E(|\psi\rangle) = -\ln\left(1-q^2\right) - \frac{q^2}{(1-q^2)}\ln q^2. \tag{47}$$

In Fig. 3 we have illustrated the dependence of the estimated entanglement of the transmitted two-mode squeezed vacuum, Eq. (44), on both the fiber length and the strength of the initial squeezing. It is seen that with increasing strength of the initial squeezing [which is, according to Eq. (47), a measure of the strength of the initial entanglement] the entanglement of the transmitted light drastically decreases with the transmission length. The transmission length $l_{\rm E}$ at which the entanglement degradation has reached half of the initial entanglement is shown in Fig. 4 as a function of the mean number of initial photons $\langle n\rangle$, which is related to the squeezing parameter $|q|^2$ according to

$$q^2 = \frac{\langle n\rangle}{\langle n\rangle + 1}\,. \tag{48}$$

Figure 4 reveals that even for relatively small mean number of photons this characteristic length of entanglement degradation is much more shorter than the absorption length. Obviously, entanglement cannot be maintained when going to more macroscopic nonclassical states. Since strong squeezing (i.e., large photon number) is typically required in quantum teleportation [1], one has to find a compromise between highest possible entanglement and lowest entanglement degradation.

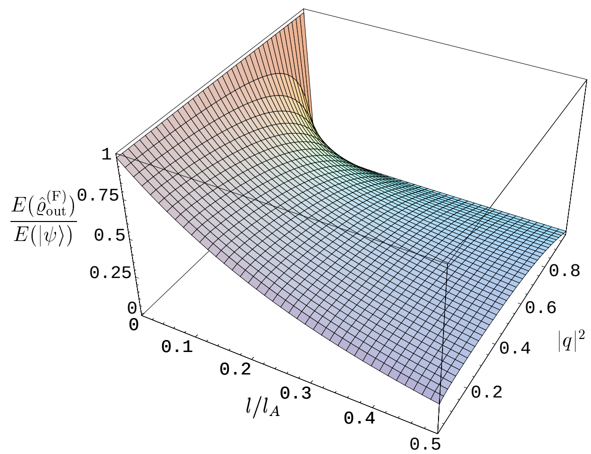

Figure 3: Estimate of the entanglement of the transmitted squeezed vacuum, Eq. (44), as a function of the initial squeezing parameter $|q|^2$ and the transmission length l [$E(|\psi\rangle)$, initial entanglement].

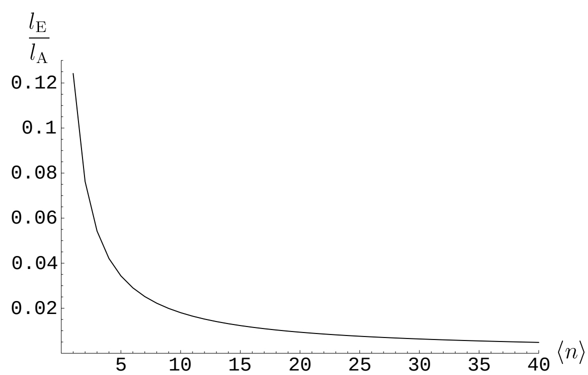

Figure 4: Entanglement degradation length l_E after which the entanglement of the transmitted two-mode squeezed vacuum has decreased to half the initial value as a function of the initial value of the mean number of photons $\langle n\rangle$ (l_A, absorption length).

5 Summary and Outlook

We have studied the entanglement degradation of a two-mode squeezed vacuum state transmitted through noisy communication channels, a typical example being the propagation along dispersing and absorbing or amplifying optical fibers. As expected, both absorption and amplification lead to entanglement degradation, because of the additional noise introduced by them. Using the quantum-optical input-output relations of radiation at dielectric four-port devices, we have derived the maximal transmission length after which an initially entangled state becomes separable. Analogously, we have found that there is a maximal gain factor for which inseparability (of any two-mode squeezed vacuum state) can be retained at all.

Knowledge of the complete quantum state at the output of the device enables us, in principle, to compute the amount of entanglement available after transmission such as the distance of the state to the set of all separable states. However, for higher-dimensional Hilbert spaces as in our case this is impossible to do in general. Therefore we have restricted ourselves to the calculation of upper bounds or estimates of upper bounds. For weak squeezing we may truncate the Hilbert space effectively at low photon numbers providing us with a way to establish an upper bound on the entanglement by exploiting the convexity property of entanglement measures. The procedure is based on the extraction of a single pure state from the output state leaving behind only separable states. For the general case of infinite dimensional Hilbert space we have derived an estimate of the upper bound.

Still, we are left with (estimates) of bounds on the entanglement. Future works must surely contain algorithms for computation of the entanglement, at least for Gaussian states. A possible step in that direction would include the calculation of the distance of a given Gaussian state to the set of all separable Gaussian states whose surface can again be parametrized by the Peres–Horodecki separability criterion.

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